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METHOD OF UNDETERMINED COEFFICIENTS IN SOLVING THIRD-ORDER LINEAR DIFFERENTIAL EQUATIONS: A COMPREHENSIVE ANALYSIS

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Highlights:

- *The method of undetermined coefficients requires that $F(x)$ has functions that have a finite family of derivatives*
- *Repeated roots in the auxiliary equation require adjusting the trial solution by multiplying by x to maintain linear independence.*
- *The undetermined coefficients can be determined through a series of differentiation and algebra.*

Abstract: Higher-order linear differential equations have great significance theoretically and practically. This paper solves a third-order linear differential equation with constant coefficients by method of undetermined coefficients. Its strengths and limitations were analysed. Analysis suggests that the method of undetermined coefficients requires that the inhomogeneous term be algebraic, exponential, or sinusoidal functions only. Additionally, if the inhomogeneous term is a solution of the differential equation itself, y_p may yield a linear combination of y_c. In such case, the prediction for y_p will be modified by multiplying the variable x however many times necessary.

Keywords: linear independence; homogeneous; linear combination; third-order; solution

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1. Introduction

Linear differential equations are fundamental mathematical tools used to model various phenomena in science, engineering, and other disciplines. They describe relationships between a function and its derivatives, where the function and its derivatives are multiplied by coefficients that are constants or functions of the independent variable.

Linear differential equations are characterized by their linearity, meaning that the dependent variable along with its derivatives are present only to the first degree and are not multiplied together or divided. This linearity allows for the use of superposition, where the addition of any two solutions to a linear differential equation also results in a solution (Boyce & DiPrima, 2017). This property greatly simplifies the solution process and enables the analysis of complex systems (Farlow, 2016).

Higher-order linear differential equations are a specific type of linear differential equation where the highest order derivative is of order n. These equations are of particular interest as they involve more complex relationships between the function and its derivatives. The highest order derivative present in a differential equation determines its order (Kreyszig, 2018).

Differential equations are categorized into partial differential equations (PDE) or ordinary differential equations (ODE) based on the presence or absence of partial derivatives. The order of a differential equation is determined by the highest order derivative it contains. A solution, or particular solution, of a differential equation of order n is a function defined and differentiable n times over a domain D. This function satisfies the given differential equation when substituted along with its n derivatives, and this holds true for every point within the domain D (Differential Equations I MATB44H3F, 2011).

A solution containing arbitrary constants corresponding to the differential equation's order is known as a general solution. On the other hand, a solution devoid of arbitrary constants is referred to as a particular solution (Hilbert, 2013).

Higher-order linear differential equations have great significance theoretically and practically. They are typically used in a variety of applications in Science and Engineering (Ross, 2021, p. 110). Differential equations find applications in physics, biology, economics, and many other disciplines, playing a crucial role in predicting and analysing the behaviour of complex phenomena (Strogatz, 2014).

Most real-world equations are second-order, though higher-order ones do show up now and then. This leads to the common belief that the world operates on a "second-order" basis in modern physics. Essentially, the key results for higher-order linear ODEs are quite similar to those for second-order equations, just with "n" replacing "2" (Lebl, 2013).

The method of undetermined coefficients is a focal point of this study due to its systematic and straightforward approach to solving nonhomogeneous linear differential equations with constant coefficients. It is characterized by the fact that its forcing function is a solution of the differential equation itself (Cook & Cook, 2022). It contains unknown constants called the undetermined coefficients which will be determined through a series of differentiation and algebra (Brauer, 1966; Nagy, 2015). This method proves highly effective for finding particular solutions when the non-homogeneous term is a polynomial, exponential, sine, or cosine function, which is frequently encountered in various practical applications. Its simplicity and directness ensure accessibility and reliability, providing clear steps and minimizing the potential for error compared to more complex techniques. Mastery of this method enables efficient handling of a wide range of differential equations, thus significantly enhancing problem-solving capabilities.

1.1. Research Objectives

This paper focuses on linear differential equations with constant coefficients, a common type of differential equation in various applications. Explicit methods available for solving these equations include the method of undetermined coefficients, exponential shift, reduction of order, and variation of parameters. Being a relatively simple solution method requiring only skills in differentiation and algebra, this paper aims to comprehensively analyse the method of undetermined coefficients. More specifically, it aims to:

1. Utilize the method of undetermined coefficients to solve a third-order linear differential equation.

2. Analyse the strengths, limitations, and versatility of the method of undetermined coefficients in solving linear differential equations.

3. Determine the family of functions for the inhomogeneous term that can be effectively addressed by the method of undetermined coefficients.

1.2. Underlying Principles of the Higher-Order Linear Differential Equation

The general linear equation of nth order can be written

$$b_o(x)\frac{d^n y}{dx^n} + b_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots b_{n-1}(x)\frac{dy}{dx} + b_n(x)y = F(x) \tag{1}$$

An equation qualifies as a homogeneous linear differential equation when the function $F(x)$ equals zero for all x. If $F(x)$ is non-zero for any x, the equation is considered non-homogeneous (Rainville & Bedient, 1989).

If the solutions of Equation 1 are y_1, y_2, and y_n and if c_1, c_2, and c_n are constants, then

$$y = c_1 y_1 + c_2 y_2 + \cdots c_n y_n \tag{2}$$

For a solution to a higher-order linear differential equation be valid, it must have linear independence. For a set of functions to be linearly independent, scalars $v_1, v_2, and\ v_n$ of Equation 3 should all be 0 (Xu, 2011).

$$v_1 y_1 + v_2 y_2 + \cdots + v_n y_n = 0 \tag{3}$$

Linear independence is crucial in determining the general solution to a differential equation, as it guarantees that the solution is not redundant (Coddington & Levinson, 1955).

Moreover, their Wronskian must not be equal to zero. The Wronskian can be obtained by getting the determinant of a square matrix with the original functions on the first row and its consecutive derivatives on the following rows.

$$W = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y'_1 & y'_2 & \cdots & y'_n \\ \vdots & \vdots & \ddots & \vdots \\ {y_1}^{(n-1)} & {y_2}^{(n-1)} & \cdots & {y_n}^{(n-1)} \end{vmatrix} \tag{4}$$

If $W = 0$, then $y_1, y_2, and\ y_n$ are considered to be linearly dependent. If $W \neq 0$, it can be deduced that they are linearly independent.

1.3. Differential Operators

Let D denote differentiation with respect to x. Then, D^k, as shown in Equation 5, refers to differentiating k times with respect to x. This is true for positive integral k.

$$D^k y = \frac{d^k y}{dx^k} \tag{5}$$

Equation 6

$$A = aD^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n \tag{6}$$

is referred to as a differential operator of n^{th} order. When this operator is applied to any function y, it produces Equation 7.

$$Ay = a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y \tag{7}$$

The coefficients $a_0, a_1, \ldots, a_n$ in the operator A may be constants or functions of x.

1.4. Differential Operators Properties

For constant m and positive integral k

$$D^k e^{mx} = m^k e^{mx} \tag{8}$$

The effect of an operator upon e^{mx} can be determined. Let $f(D)$ be a polynomial in D,

$$f(D) = a_0D^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_n. \tag{9}$$

Then

$$f(D)e^{mx} = a_0m^ne^{mx} + a_1m^{n-1}e^{mx} + \cdots + a_{n-1}me^{mx} + a_ne^{mx}, \tag{10}$$

Therefore,

$$f(D)e^{mx} = e^{mx}f(m). \tag{11}$$

If m satisfies the equation $f(m) = 0$, then in light of Equation 11,

$$f(D)e^{mx} = 0 \tag{12}$$

Equation 13 and Equation 14 demonstrate how the operator $D - a$ affects the product of a function y and e^{ax}.

$$(D - a)(e^{ax}y) = D(e^{ax}y) - ae^{ax}y \tag{13}$$

$$= e^{ax}Dy \tag{14}$$

Subsequently, the use of the operator $(D - a)^2$ is shown on Equation 15 and Equation 16.

$$(D - a)^2(e^{ax}y) = (D - a)(e^{ax}\mathrm{Dy}) \tag{15}$$

$$= e^{ax}D^2y \tag{16}$$

Repeating the process, the effect of the operator $(D - a)^n$ leads to:

$$(D - a)^n(e^{ax}y) = e^{ax}D^ny \tag{17}$$

By linearity of differential operators, it can be concluded that when $f(D)$ represents a polynomial in D, then

$$e^{ax}f(D)y = f(D - a)[e^{ax}y] \tag{18}$$

1.5. The Auxiliary Equation Yielding Distinct Roots

Linear Homogeneous Equations that have constant coefficients,

$$a_0\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots a_{n-1}\frac{dy}{dx} + a_ny = 0 \tag{19}$$

can be rewritten as

$$f(D)y = 0 \tag{20}$$

where $f(D)$ is a linear differential operator. If the auxiliary equation $f(m) = 0$ has any root m, then

$$f(D)e^{mx} = 0 \tag{21}$$

It means that $y = e^{mx}$ is a solution of Equation 20.

$$f(m) = 0 \tag{22}$$

Equation 22 is called the auxiliary equation corresponding to Equation 19 and Equation 20.

If the auxiliary equation of Equation 19 has distinct roots $m_1, m_2, \ldots, m_n$, then its general solution can be written as

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_3 e^{m_n x}. \tag{23}$$

Here, c_1, c_2, and c_n are arbitrary constants. Moreover, the functions corresponding to them are linearly independent.

1.6. Auxiliary Equation Yielding Repeated Roots

Assume that in Equation 20, the operator $f(D)$ contains factors that repeat. That means that the auxiliary equation shown in Equation 22 has repeated roots. Thus, the method used in Section 1.5 does not produce the general solution. Let the auxiliary equation possess three roots that are equal: $m_1 = b, m_2 = b$, and $m_3 = b$. The solution this will yield is:

$$y = c_1 e^{bx} + c_2 e^{bx} + c_3 e^{bx} \tag{24}$$

$$y = (c_1 + c_2 + c_3)e^{bx} \tag{25}$$

Equation 25 can be rewritten as

$$y = c_4 e^{bx} \tag{26}$$

where $c_4 = c_1 + c_2 + c_3$. Thus, this method only produced one solution. The difficulty exists because the three solutions that correspond to $m_1 = m_2 = m_3 = b$ are linearly dependent.

A method to obtain an n number of linearly independent solutions that correspond to n identical roots of the auxiliary equation is required. Suppose that Equation 22 has n equal roots

$$m_1 = m_2 = \cdots = m_n \tag{27}$$

Therefore, the operator $f(D)$ should have a factor $(D - b)^n$. Multiple linearly independent y's must be found for which

$$(D - b)^n y = 0 \tag{28}$$

In Equation 18, we supposed that $f(D) = D^n$ and $y = x^k$. We can obtain Equation 29 by using the Exponential Shift.

$$(D-m)^n(x^k e^{mx}) = e^{mx} D^n x^k \tag{29}$$

However, $D^n x^k = 0$ for $k = 0, 1, 2, \dots, n-1$, which results to Equation 30.

$$(D-m)^n(x^k e^{mx}) = 0 \quad for\ k = 0, 1, 2, \dots, (n-1) \tag{30}$$

Because of the existence of repeated roots, $m = b$. As a result, Equation 30 can be also be written as

$$(D-b)^n(x^k e^{bx}) = 0 \quad for\ k = 0, 1, 2, \dots, (n-1) \tag{31}$$

The functions $y_k = x^k e^{bx}$ with $k = 0, 1, 2, \dots, (n-1)$ are linearly independent because each function includes a distinct power of x multiplied by e^{ax}, ensuring no linear combination of these functions can be simplified to zero unless all coefficients are zero.

The general solution of Equation 28 is

$$y = c_1 e^{bx} + c_2 x e^{bx} + \cdots + c_n x^{n-1} e^{bx} \tag{32}$$

Moreover, if $f(D)$ has the factor $(D-b)^n$, then Equation 28 can take the form

$$g(D)(D-b)^n y = 0. \tag{33}$$

Here, $g(D)$ includes every factor of $f(D)$ except $(D-b)^n$. Then any of the solutions of Equation 28 also satisfies Equation 33, and therefore, Equation 20.

For each root m_i of the auxiliary equation, which may either be distinct or among a set of identical roots, there exists a corresponding solution

$$y_i = c_i e^{m_i x}. \tag{34}$$

Meanwhile, for n equal roots $m_1, m_2, \dots, m_n$, each equal to b, the corresponding solutions are

$$y = c_1 e^{bx}, c_2 x e^{bx}, \dots, c_n x^{n-1} e^{bx} \tag{35}$$

The collection of solutions in Equation 35 includes the appropriate number of elements, which corresponds to the order of the differential equation. For every root of the auxiliary equation, there is a corresponding solution. The obtained solutions are therefore linearly independent (Rainville & Bedient, 1989).

Determining whether a solution exists and whether it is valid on its own is one of the main issues when solving differential equations. At least one solution must satisfy the equation and the beginning circumstances in order for "existence" to be established. This is guaranteed by the equation's components' consistency and smoothness. On the other hand, "Uniqueness" rules out other solutions by claiming that only one fits these requirements. According to Lebl, these foundational ideas are important because they guarantee that solutions obtained from mathematical models faithfully represent real-world situations and provide consistent and trustworthy predictions (Lebl, 2013).

1.7. Underlying Principles of Method of Undetermined Coefficients

Section 1.7 discusses the theoretical background of the method of undetermined coefficients is discussed. Rainville & Bedient (1989) examined the principles behind this method.

Consider $f(D)$ as a polynomial in the differential operator D. Note the equation

$$f(D)y = F(x) \tag{36}$$

The roots of the auxiliary equation $f(m) = 0$ are denoted as

$$m = m_1, m_2, \dots, m_n \tag{37}$$

Equation 38 provides the general solution

$$y = y_c + y_p \tag{38}$$

where y_c is determined using the roots m in Equation 37 and y_p denotes a particular solution of Equation 36 that remains to be determined.

Suppose $F(x)$, the right-hand side of Equation 36, is a particular solution of a homogeneous linear differential equation with constant coefficients:

$$g(D)F = 0 \tag{39}$$

and its auxiliary equation yields the roots

$$m' = m'_1, m'_2, \dots, m'_k \tag{40}$$

Note that the values in Equation 40 can be determined by inspecting $F(x)$.

Consider the differential equation

$$g(D)f(D)y = 0. \tag{41}$$

Equation 41 has its roots in the values of m from Equation 37 and m' obtained from Equation 40 through their respective auxiliary equations. Thus, the general solution of Equation 41 contains the y_c from Equation 38. Therefore, it is of the form

$$y = y_c + y_q \tag{42}$$

Additionally, any particular solution of Equation 36 should also satisfy Equation 41. If

$$f(D)\left(y_c + y_q\right) = F(x), \tag{43}$$

then $f(D)y_q = F(x)$ since $f(D)y_c = 0$. After removing y_c from Equation 41, a function y_q is obtained whose coefficients can be adjusted to satisfy Equation 36 for specific numerical values. By appropriately determining the coefficients in y_q, it can be ensured that y_q equals y_p. This determination of coefficients can be accomplished through straightforward algebraic methods.

1.8. Allowable Functions

The method of undetermined coefficients relies on making an informed assumption about the structure of the particular solution. **Table 1** displays the general form of the right member $F(x)$ along with their corresponding prediction for the particular solution. The specific functions that this method can handle are those that have a finite family of derivatives.

Table 1. Functions suitable for method of undetermined coefficients

F(x)	**y_p**
ae^{mx}	Ae^{mx}
a cos(βx)	A cos(βx) + B sin (βx)
b sin (βx)	A cos(βx) + B sin (βx)
a cos(βx) + b sin (βx)	A cos(βx) + B sin (βx)
n^{th} degree polynomial	$A_nx^n + A_{n-1}x^{n-1} + \ldots + A_1x + A_0$
where, A,B,a,b are arbitrary constants	

2. Methodology

This section solves one higher-order linear differential equation using undetermined coefficients method.

Consider the third-order linear nonhomogeneous differential equation

$$y''' - 3y'' + 3y' - y = xe^x \tag{44}$$

It has an auxiliary equation

$$m^3 - 3m^2 + 3m - 1 = 0.$$

Notice that this can be simplified and rewritten as

$$(m-1)^3 = 0$$

$$(m-1)(m-1)(m-1)=0$$

Equating each factor to 0, we obtain the roots. Therefore, the roots of Equation 44 are $m = 1,1,1$.

The general solution of Equation 44 follows the form of Equation 38.

Since the auxiliary equation yields repeated roots, y_c follows the form of Equation 23.

$$y_c = c_1 e^{m_1 x} + c_2 x e^{m_2 x} + c_3 x^2 e^{m_3 x}$$

Since $m_1 = m_2 = m_3 = 1$,

$$y_c = c_1 e^x + c_2 x e^x + c_3 x^2 e^x \tag{45}$$

Equation 44 features a right-hand side composed of an exponential function and an algebraic function multiplied together. Both functions have a limited number of derivatives, thus allowing for the application of the method of undetermined coefficients.

Initial assumption for y_p is

$$y_p = e^x(Ax + B)$$

$$y_p = Axe^x + Be^x \tag{46}$$

However, Equation 46 appears to be a linear combination of Equation 45. Therefore, Equation 46 is not a valid guess for y_p. To ensure linear independence, the variable x may be multiplied to the predicted y_p to prevent a linear combination of y_c.

$$y_p = [e^x(Ax + B)]x^3$$

results to

$$y_p = Ax^4 e^x + Bx^3 e^x \tag{47}$$

Equation 47 is the appropriate prediction for y_p because it does not yield a linear combination.

Since Equation 44 is a third-order differential equation, y_p must be differentiated three times.

$$y_p' = e^x(Ax^4 + 4Ax^3 + Bx^3 + 3Bx^2) \tag{48}$$

$$y_p'' = e^x(Ax^4 + 8Ax^3 + Bx^3 + 6\text{Bx}^2 + 12Ax^2 + 6\text{Bx}) \tag{49}$$

$$y_p''' = e^x(Ax^4 + Bx^3 + 12Ax^3 + 36Ax^2 + 9Bx^2 + 18\text{Bx} + 24\text{Ax} + 6\text{B}) \tag{50}$$

Equations 47, 48, 49, and 50 are substituted to Equation 44, which yields;

$$y_p''' - 3y_p'' + 3y_p' - y_p = xe^x$$

$$e^x(Ax^4 + Bx^3 + 12Ax^3 + 36Ax^2 + 9Bx^2 + 18Bx + 24Ax + 6B) - 3e^x(Ax^4 + 8Ax^3 + Bx^3 + 6Bx^2 + 12Ax^2 + 6Bx) + 3e^x(Ax^4 + 4Ax^3 + Bx^3 + 3Bx^2) - e^x(Ax^4 + Bx^3) = xe^x$$

which will simplify to

$$24Ax + 6B = x$$

Equating the coefficients of the linearly independent terms:

$$x: 24A = 1$$

$$A = \frac{1}{24}$$

$$x^0: 6B = 0$$

$$B = 0$$

Substituting the obtained values of A and B into Equation 47,

$$y_p = \frac{1}{24}x^4e^x \quad (51)$$

Since

$$y = y_c + y_p,$$

the general solution of Equation 44 is the linearly independent

$$y = c_1e^x + c_2xe^x + c_3x^2e^x + \frac{1}{24}x^4e^x.$$

An advantage of this method is it makes the problem solvable by simple algebra (Agarana & Akinlabi, 2019).

3. Results and Discussion

The method of undetermined coefficients is a simple method, only requiring skills in differentiation and algebra. This makes it an accessible approach for those with a basic understanding of differential equations (Simmons & Krantz, 2007).

It works for differential equations with constant coefficients. However, its use is limited because it requires that $F(x)$ has functions that have a finite family of derivatives. This is only applicable to algebraic, sinusoidal, and exponential functions. The functions mentioned above may sometimes be products of one another (University of Alabama in Huntsville, 2019). This is a necessary requirement because solving for y_p involves predicting its form, and this is only feasible for functions whose derivatives have predictable patterns.

If $F(x)$ consists of products that include an exponential function, the general rule is to first remove the exponential component and write down the guess for the remaining portion of the function. After this, the exponential part is reattached without any leading coefficient (Dawkins, 2022).

Additionally, when guessing for y_p, one may encounter difficulties when predicting its appropriate form. This happens when $F(x)$ is a solution of the differential equation itself. The initial guess for y_p may yield a linear combination of y_c, such as the case in Section 2. In such situations, the variable x is multiplied to the initial guess for y_p however many times necessary to prevent a linear combination. The University of Utah (2021) calls this the 'fixup rule'. If the initial trial solution produces duplicates of y_c, then it has to be multiplied by x until it is no longer a duplicate solution. This is a disadvantage of this method, as this issue can complicate the solution process. Thus, it requires careful consideration (Apostol, 1969).

4. Conclusion

While the method of undetermined coefficients is revered for its simplicity and accessibility, its effectiveness depends on the existence of finite families of derivatives of the inhomogeneous term. Moreover, it demands a cautious approach in possible cases of linear dependence in the solution. Additionally, while it is a powerful tool for solving a specific class of higher-order linear differential equations, it also poses certain challenges. A higher-order differential equation may involve more complex algebra and differentiation. Therefore, determining if the method of undetermined coefficients is suitable for a given differential equation is crucial for ease and accuracy of solution.

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Conflicts of Interest

The authors declare no conflict of interest.

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